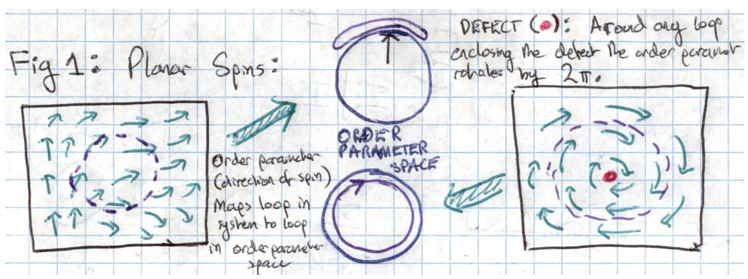
Homotopy in Condensed Matter

William Dorrell - December 2018 - PHYS 268R Research Project

Ideas from Topology have found widespread use within condensed matter physics, as recognised by the 2016 Physics Nobel Prize. Here we shall discuss the use of homotopy to categorise and analyse defects in ordered media. This article draws almost entirely from the fantastic review by N. D. Mermin "Topological Theory of Defects in Ordered Media" with mathematical support from M. Nakahara's "Geometry, Topology & Physics." Perhaps a dusting of any group theory the reader happens to have would also be useful.

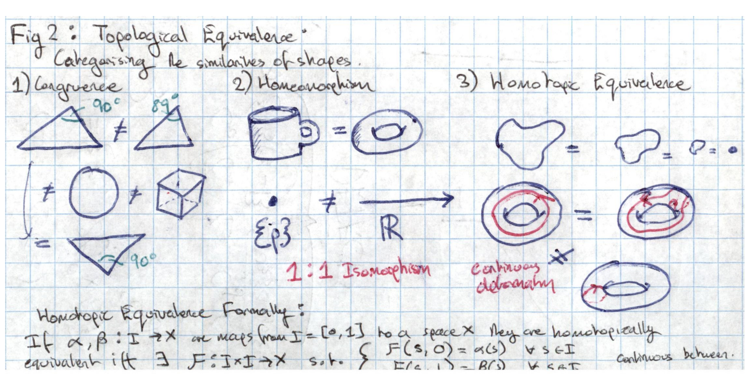
Ordered Media

An order parameter is a variable that ranges from 0 in a disordered phase to a finite value in an ordered phase, representing a spontaneously broken symmetry. It is therefore a mapping that associates every point in an ordered medium with a point in order parameter space - fig 1, we will require that this mapping is continuous at all points except perhaps singular defects. We shall consider defects whose dimensionality is two less than that of the medium (point defects in 2D, line defects in 3D) - an analysis very similar to ours will permit study of defects of any dimensionality, however there is not time to discuss such topics here. These defects can have noticeable effects on the medium at arbitrarily large distances - fig 1. In order to begin measuring, classifying and analysing such defects we will draw a loop around the defect; this loop in the medium will be mapped to a loop in order parameter space and it is in examining these loops that topology can aid our understanding.



Topology & Homotopy

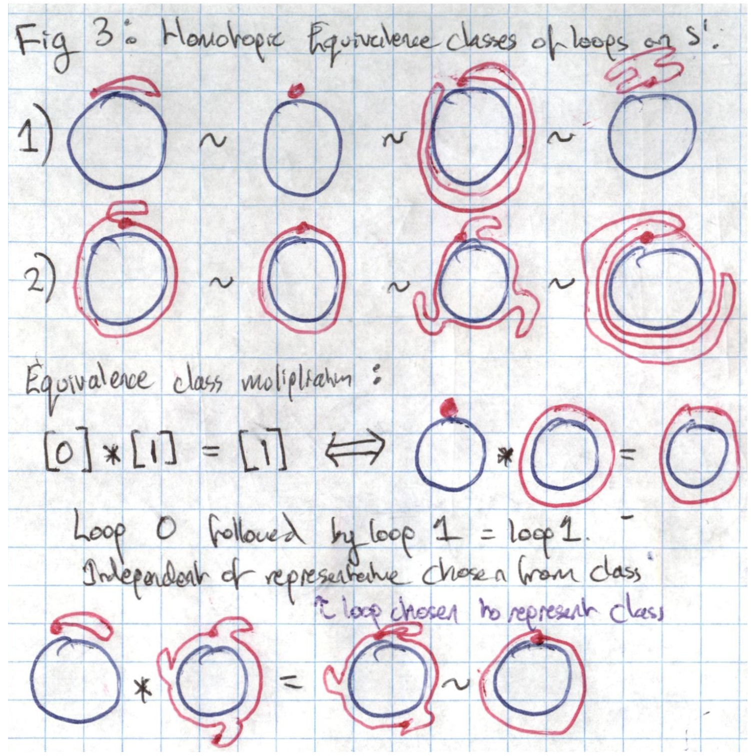
Topology concerns itself with the classification of shapes (or topological spaces more broadly). There are however many possible ways one might want to define an equivalence between two shapes - fig 2, for example one could require that they have all the same angles and lengths, known as congruence. This however is a very restrictive definition of similar, there is an obvious way in which some shapes are more similar than others without exact congruence. Famously coffee mugs and doughnuts may be considered identical if one defines identicality as a homeomorphic equivalence - i.e. one may continuously deform either shape into the other via a 1:1 correspondence. We shall be using a slightly weaker version of equivalence called homotopy which equates two shapes if they can be continuously deformed into one and other. To highlight the difference, the real line and a point are homotopic equivalents but not homeomorphic; we can deform the real line into a point ($\therefore$ homotopic equivalents), however there is no 1:1 mapping between a point and the real line ($\therefore$ not homeomorphic). We shall use the homotopic equivalence of loops in order parameter space as a handle to understand defects in ordered media.



First Fundamental Group

Given a topological space, X, and a point, x_0, we can define homotopic equivalence classes of loops which begin and end at x_0, fig 3. It turns out that these equivalence classes, in conjunction with the combination rule in which the product of two classes comprises the set of loops formed by traversing a loop from one class followed by one from the second - fig 3, form a group, called the first fundamental group based at x_0, denoted $\pi_1(X, x_0)$. We would rather progress with the first fundamental group of the space without having to specify a base, $\pi_1(X)$, however there is a small subtlety in relating fundamental groups based at different points. We distinguish between Abelian and Non-Abelian spaces, fig 5. For Abelian spaces there is an ismorphism between $\pi_1(X, x_0)$ and $\pi_1(X, y_0)$ where $x_0, y_0 \in X$ and as such one may ignore the base point allowing us to establish an isomorphism between classes of freely homotopic loops and $\pi_1(X)$.

On the other hand, Non-Abelian spaces, as illustrated in fig 4, do not possess such a simple isomorphism. There are, in general, more elements of $\pi_1(X, x_0)$ than there are classes of freely homotopic loops. Further the choice of x_0 may change the behaviour of $\pi_1(X, x_0)$. Figure 4 gives a rough illustration of the route round this problem, we construct an isomorphism between conjugacy classes of the fundamental group based at a point and categories of loops. This permits us to perform the same analysis albeit with a small additional complication.

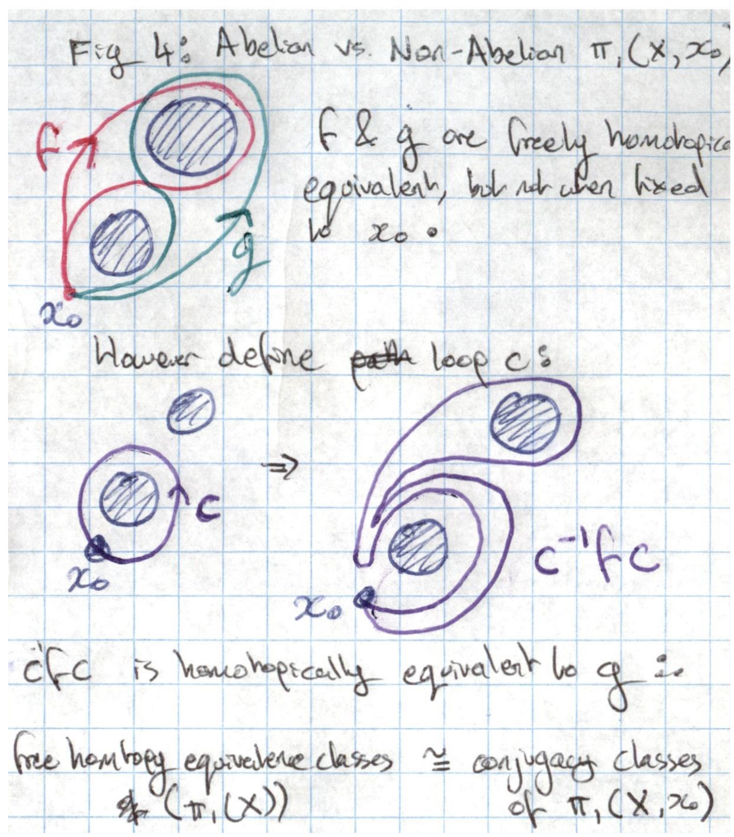


Fundamental Theorem of the Fundamental Group

We shall quote without proof the following result. Take your order parameter space and choose an arbitrary reference order parameter, f. Now take a group G that satisfies two conditions: first G is continuous, second Gf, the group action of G on f, maps f to all parts of order parameter space. Now create a second group, H, which contains all the elements of G that map f onto itself (If $g \in G$ s.t. $gf = f$ then $g \in H$), this is the isotropy subgroup. One can see, after a little squinting if your group theory is rusty, that the space, X, is isomorphic to $\frac{G}{H}$ the coset group of H in G. Further let H_0 be the set of elements of H that can be continuously mapped to indetity. Then:

$$\frac{H}{H_0} \cong \pi_1\left(\frac{G}{H}\right) \cong \pi_1(X)$$

This may currently seem extraordinarily obtuse, a couple of examples will illustrate how simple this scheme in fact is.]



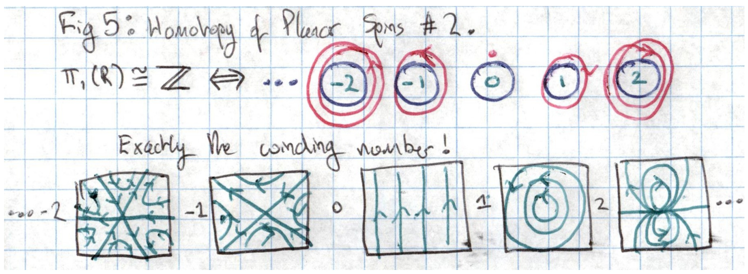
Point Defects in Planar Spins

The order parameter of planar spins may be any value $[0, 2\pi]$, meaning our order parameter space $R = S^1$, fig 1. Forming a group G that maps a reference order parameter, say 0, to $[0, 2\pi]$ we might think to choose $SO(2)$. However $SO(2)$ is not continuous. Luckily a theorem allows one always to create a cover of any group that is continuous, called the universal cover. The universal cover of $SO(2)$ is $T(1)$, the group of 1D translations, so lets choose $G = T(1)$. We must now extract the isotropy subgroup H, in this case it is obvious this is $\{2\pi n\}\ \forall\, n \in \mathbb{Z}$. Our final step requires the creation of H_0, the set of elements of H continusouly connected to identity, as H is discrete H_0 simply contains the identity.

Finally this allows us to say the following:

$$\pi_1(R) \cong \pi_1\left(\frac{G}{H}\right) \cong \frac{H}{H_0} \cong H \cong \mathbb{Z}$$

To further interpret, this simply argues that defects in a system of planar spins, which are isomorphic to homotopy equivalence classes of loops on S^1 - the order parameter space, may be labelled by an integer quantifying the number of complete cycles round the circle the loop completes. As illustrated in figure 5, this is simply the winding number!

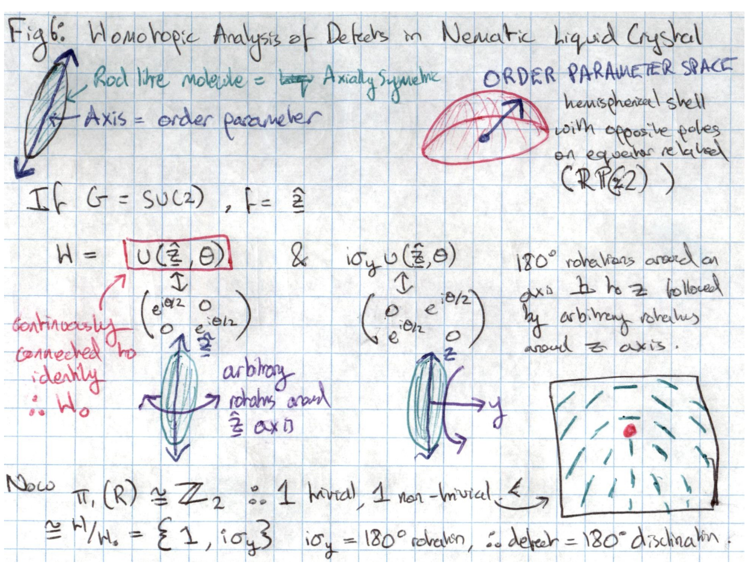


Line Defects in Nematic Liquid Crystals

Nematic Liquid Crystals present a slightly more complicated example. They are firstly three dimensional, this means we shall consider line defects such that a loop is still sufficient to enclose the defect. The order parameter can be described as point on a spherical shell with opposite poles associated, fig 6. Again the natural first group to consider that maps a reference order parameter f to all other order parameters would be $SO(3)$, the group of rotations, we however use the (continuous) universal cover of $SO(3)$, $SU(2)$. The isotropy subgroup of this, H, contains rotations by any angle around the axis of orientation of the rod molecule and rotations of 180° around an axis perpendicular to the orientation of the molecule (fig 6). Of these only the first category are continuously connected to identity and are therefore members of H_0. This leads to:

$$\pi_1(R) \cong \pi_1\left(\frac{G}{H}\right) \cong \frac{H}{H_0} \cong \mathbb{Z}_2$$

This is a useful result, it says in nematic liquid crystals there are two types of line defects, one trivial (which can be removed by only local rearrangement of molecules) and one topologically non-trivial - fig 6.



Non-Abelian Fundamental Groups & Applications to Biaxial Nematic Liquid Crystals

We previously discussed that, for a Non-Abelian fundamental group, there was an isomorphism between *conjugacy classes* of the fundamental group and defects (or equivalently freely homotopic loops). Biaxial Nematic liquid crystals are just such a Non-Abelian system (and recently experimentally realised!). They are similar to the standard nematic liquid crystal except rather than a axially symmetric rod it can be imagined as a cuboid with three distinct side lengths, fig 7.

Just to follow through the reasoning one final time: we might create a group G, that maps a reference order parameter to any other, and equate it with $SO(3)$. Within this context H, the isotropy subgroup of G, would be only 180° rotations about three orthogonal axis, fig 7. This makes H a four member group - identity and the three rotations. If we now change to considering the universal cover of G, $SU(2)$, we must concurrently change H. The $SU(2) : SO(3)$ mapping is 2:1 since members of SU(2) with opposite sign are mapped to the same element of SO(3). Therefore the isotropy subgroup of $G' = SU(2)$ is H' which contains all the members of H twice, once multiplied by -1. This, it turns out, is nothing but the Quaternion group - a Non-Abelian group as suggested. This group is discrete $\frac{H}{H_0} = H$ so we must find our defects by relating them not to elements of this quanternion group but to conjugacy classes of the group. Figure 7 highlights these (there are five) and we come to our final question. What happens when two defects in this medium intersect?

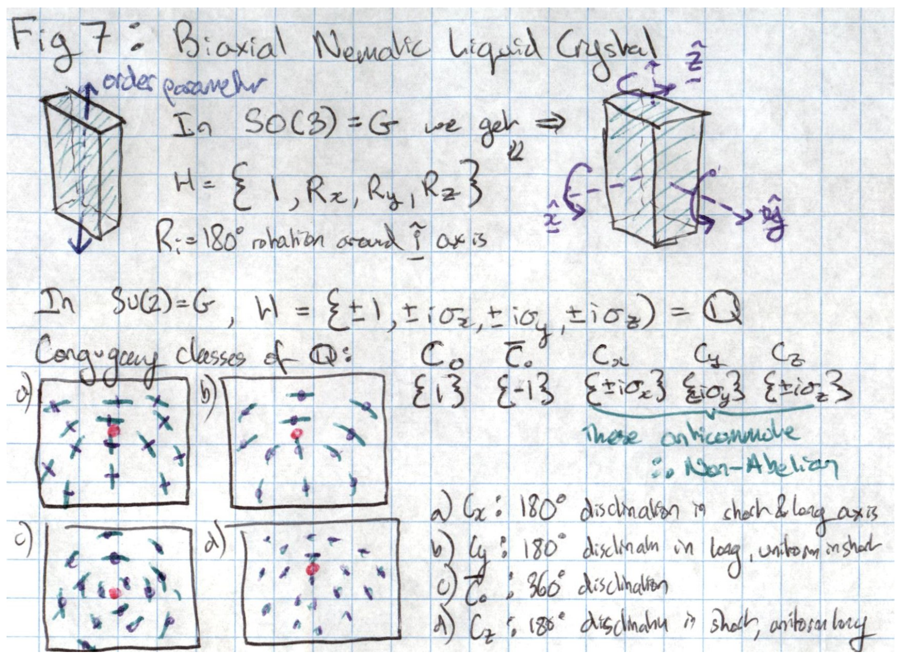


Fig 7: Biaxial Nematic Liquid Crystal

Figure 8 show such an intersection. It also illustrates how one may measure whether or not the trailing link that is left behind is trivial or a defect via the conjugacy class of the loop drawn around it. Next a series of clever deformations allows you to convert the loop into a series of measurements of the two lines, namely $\beta \circ \alpha \circ \beta^{-1} \circ \alpha^{-1}$ where α and β are the conjugacy classes of the two lines. In the Quaternion group this combination always produces either 1 representing a trivial defect or -1 if the two intersecting lines are different disclinations by 180°.

To conclude defect lines in biaxial nematic crystals may pass through each other unimpeded, except when they are two different members of $\{C_x, C_y, C_z\}$. In this case the two lines are forever linked by a trailing scar of type $\overline{C_0}$. As Mermin says: "Arriving at these conclusions without the aid of homotopy groups requires a higher order of geometrical imagination than I, at least, possess; I commend them to the attention of those who suspect that the use of homotopy groups simply obscures with intricate and arid formalism what would otherwise be intuitively clear." Fair play topology!

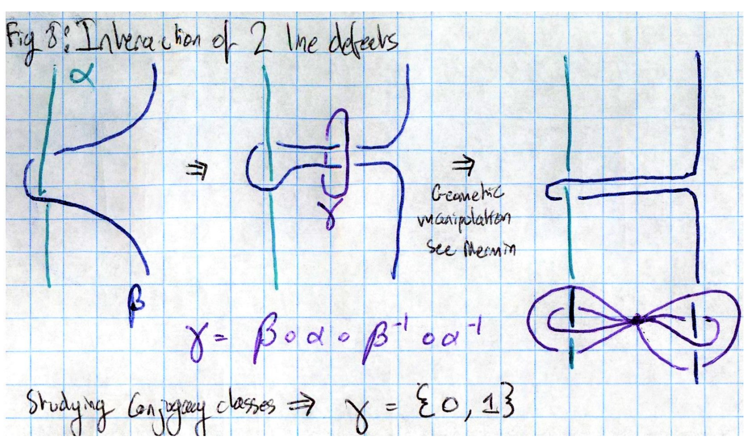


Fig 8: Interaction of 2 line defects

Final Remarks

I hope I have illustrated the surprising insights offered by what at first seems arcane formalism. Defects play a key role in defining material properties; as such a slicker formalism for their comprehension could provide a handle for better explaining and utilising condensed matter systems. On a broader point this is a beautiful case of pretty mathematics being powerfully put to use in physical systems, a situation that reflects well on both the physics and maths involved.